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Buck

Some Graphical Solutions of Electric Railway Problems



**SOME GRAPHICAL SOLUTIONS OF ELECTRIC  
RAILWAY PROBLEMS**

**BY**

**ALONZO MORRIS BUCK**  
**M. E. Cornell University, 1904**

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**THESIS**

**Submitted in Partial Fulfillment of the Requirements for the**

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**IN**

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RAILWAY PROBLEMS

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## SOME GRAPHICAL SOLUTIONS OF ELECTRIC RAILWAY PROBLEMS

### I. INTRODUCTION

In the solution of railway problems involving the characteristics of the motive power it is difficult to use analytical methods, principally because it is impossible to obtain a satisfactory general equation for the curves of an engine or motor of any specified type. The relation between speed and tractive effort, for instance, is so involved that any attempt to obtain a formula leads to assumptions which cannot be made without seriously affecting the accuracy of the final result.\* This is true not only of the steam locomotive, but also of the various types of electric motors ordinarily used for train propulsion.

The graphical methods, in contrast with the analytical, form an accurate and at the same time an easy means of attack applicable to any possible combination of characteristics and any range of conditions which may be met in practice. It is the purpose of this bulletin to develop a number of new graphical methods which, in connection with other well-known ones, aid materially in the solution of such problems. While most of these were developed in connection with problems of electric train performance, a number of them are equally applicable to any type of motive power, a fact which is set forth in the paragraphs which follow.

The majority of these methods were developed by the writer in connection with classroom instruction. One of the ways of obtaining motor performance with varying potential and one for finding the "effective" value of the motor current are due to Mr. S. Sekine, a graduate student in Railway Engineering in the University of Illinois, who is also responsible for a portion of the method of plotting speed-time and distance-time curves.

### II. MOTOR PERFORMANCE WITH VARYING POTENTIAL†

The performance characteristics of a railway motor are ordinarily furnished by the manufacturer for the normal potential and are usually assumed to be accurate under such conditions. Often it is desirable to find the motor performance when abnormal potential is impressed on the terminals, since in practice the line pressure is subject to wide fluctuations, and the motors are always operating at subnormal potential while the controller is being turned to the full-speed position.

\*See C. O. Mailloux, Discussion on paper by F. W. Carter, Transactions A. I. E. E., Vol. XXII, p. 165 (1903).

†For a brief discussion of this topic see Electric Railway Journal, Sept. 18, 1915.



The torque produced by a given current in a series motor is practically independent of the line pressure,\* so that recalculation of this quantity is unnecessary for any ordinary conditions of operation met with in practice, unless, of course, the field strength is purposely reduced. The only other important variable to be considered is the motor speed.

In an electric motor the applied pressure is used up in two ways; a portion overcomes the drop due to the resistance of the windings, and the remainder opposes the counter e.m.f. generated in the armature. If the field flux remains constant, the speed will vary in direct proportion to the counter e.m.f. which is developed. This may be expressed by the equation

$$\frac{V_2}{V_1} = \frac{E_2 - Ir}{E_1 - Ir} \dots \dots \dots (1)$$

in which  $V_1$  and  $V_2$  are the speeds when  $E_1$  volts and  $E_2$  volts are applied at the terminals, respectively,  $I$  is the current flowing through the armature, and  $r$  is the motor resistance, or that portion in the armature and the circuits in series therewith.

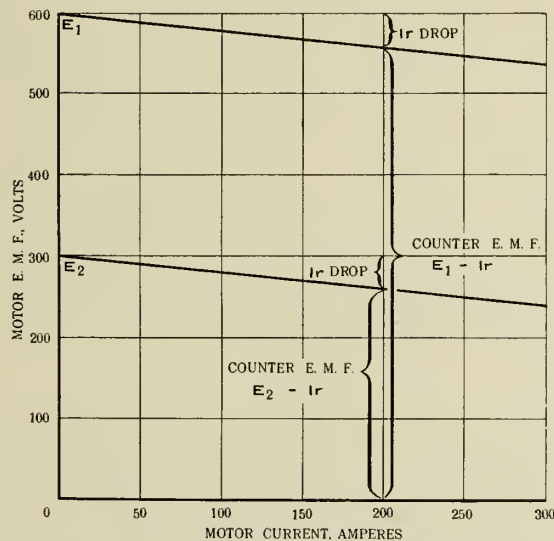


FIG. 1. VOLT-AMPERE DIAGRAM FOR ELECTRIC MOTOR.

In order to make the calculation graphically it is only necessary to determine the relative values of  $E_1 - Ir$  and  $E_2 - Ir$ , from which the ratio of speeds may be found directly. A simple method of showing the relations between these values is to construct a diagram with motor volts as ordinates and armature amperes as abscissae, as

\*A. M. Buck, The Electric Railway, p. 53.





shown in Fig. 1. Since the  $Ir$  drop is a direct function of the armature current, it can be represented for all values of current by the intercepts on a straight line with the proper slope. This may be drawn through the origin, but, since we are principally concerned with the difference between the terminal pressure and the  $Ir$  drop, it is better to draw it from the line of full pressure at the motor terminals,  $E_1$ . If the terminal pressure is then changed to  $E_2$  volts, it will not affect the slope of the  $Ir$  line, but will change its position so that it begins at the point  $E_2$ . In each case the counter e.m.f. is the residue after subtracting the  $Ir$  drop, as shown in the diagram. All that remains is to obtain a graphical relation between  $V_1$  and  $V_2$ , which is proportional to these values of counter e.m.f. Two methods of doing this have been developed.

The first method of calculation is shown in Fig. 2. Here the volt-ampere diagram of Fig. 1 is reproduced, along with the speed-current curve of the motor, as determined by test or from design calculations, the axes of current being in the same straight line. The

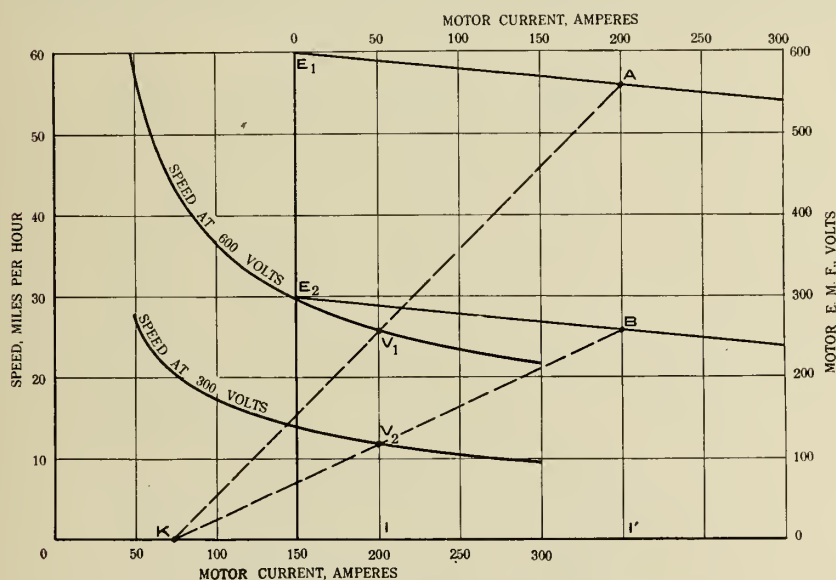


FIG. 2. CONSTRUCTION FOR OBTAINING MOTOR SPEEDS AT DIFFERENT POTENTIALS.

current scales and their positions along the axis may be chosen as desired, their relation to each other being immaterial. The speed of the motor at the terminal pressure  $E_1$  is represented by the ordinate  $V_1$ . It is desired to find the corresponding value of speed  $V_2$  at  $E_2$  volts and the same current  $I$ . Draw a line through  $A$  at the value of current  $I$  on the volt-ampere diagram and also through  $V_1$ . This





will intersect the axis of abscissae at some point  $K$ . From  $K$  draw the line  $KB$ , through the corresponding point  $B$  on the volt-ampere diagram for the same current and the new pressure  $E_2$ . This locates  $V_2$ , the speed at  $E_2$  volts, at the intersection of  $KB$  with the current ordinate. It must be correct since, by similar triangles,

$$\frac{IV_1}{IV_2} = \frac{I'A}{I'B} \dots\dots\dots (2)$$

It may be seen from Fig. 1 that  $I'A$  and  $I'B$  are the values of counter e.m.f. corresponding to the pressures  $E_1$  and  $E_2$  at the current  $I$ .

It should be noted that a different position of the point  $K$  will be located for each value of current, and in some cases it may be at too great a distance from the body of the diagram. To obviate this the relative positions of the speed-current and the volt-ampere diagrams may be changed, always keeping their current axes together.

In some cases it is preferable to make the entire construction on the speed current diagram. The arrangement for this method is shown in Fig. 3. Here the base of the volt-ampere diagram is taken the same as that for the speed-current curve, and the propor-

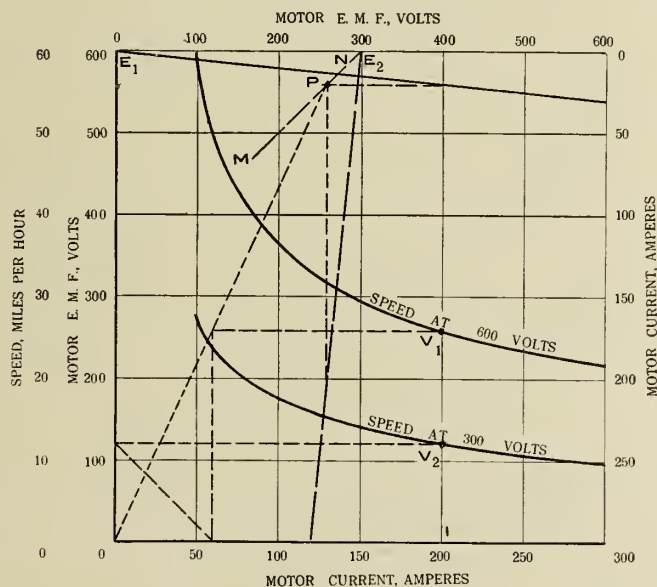


FIG. 3. SECOND METHOD FOR OBTAINING MOTOR SPEEDS AT DIFFERENT POTENTIALS.

tional division is made by swinging one set of values of counter e.m.f. through an angle of 90 degrees, so that  $E_1N$  is equal to  $OE_2$ . The two projections of the values of counter e.m.f. will meet at some point, such as  $P$ , and a line drawn connecting  $P$  with the origin will



divide the ordinate and abscissa of any point along it proportionally to these two values. Then, by projecting the speed at  $E_1$  volts onto this line, the speed at  $E_2$  volts and the same current are given by the corresponding abscissa, and may be carried back through 90 degrees and plotted on the original current ordinate, as shown.

A further inspection of Fig. 3 shows that the locus of the point  $P$  will be a line  $MN$ , which passes through  $N$ , corresponding to zero  $Ir$  drop, and makes an angle of 45 degrees with the axes. The proof of this construction is that the  $Ir$  drop is the same for a given current irrespective of the terminal pressure. For this reason it is unnecessary to swing mechanically the counter e.m.f. line through 90 degrees to locate  $P$ . Draw  $MN$  from the intersection  $N$  of the projections of  $E_1$  and  $E_2$  (taken at right angles, as explained above). Any point on the counter e.m.f. line will then give a projection on  $MN$ , as at  $P$ , thus saving the preliminary construction.

### III. MOTOR PERFORMANCE WITH RESISTANCE

To determine the performance of a motor when a resistance is inserted in series with the armature, the constructions given in Figs. 2 and 3 may be used with a slight modification. Fig. 4 is the

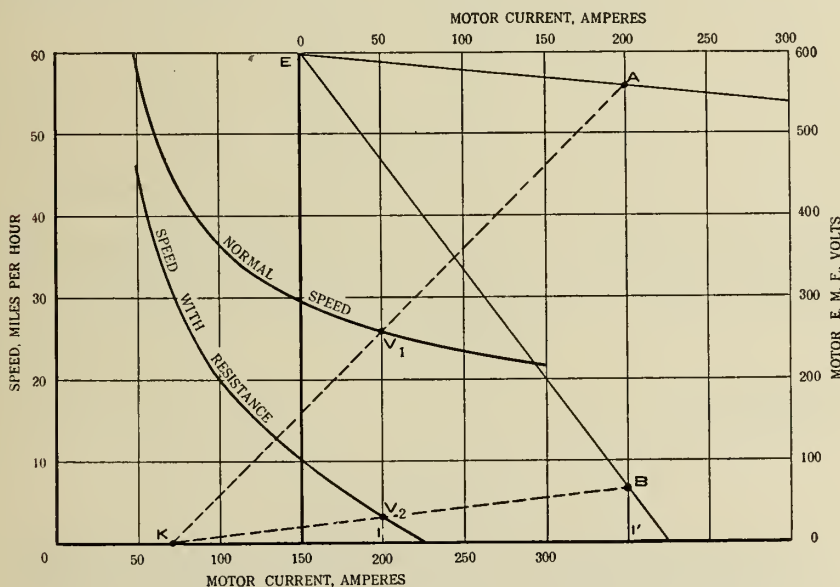


FIG. 4. FIRST METHOD FOR OBTAINING MOTOR SPEEDS WITH RESISTANCE.

same as Fig. 2, except that the  $Ir$  drop at a different pressure has been replaced by a line  $E_1B$  representing the drop  $I(R + r)$ , in which  $R$  is the external resistance in the circuit. The procedure is the



same as that explained in the determination of motor performance with varying potential, and the proof of the construction is identical.

The method of Fig. 3 can equally well be used for determining motor speeds with resistance, as shown in Fig. 5. Since the  $IR$  drop

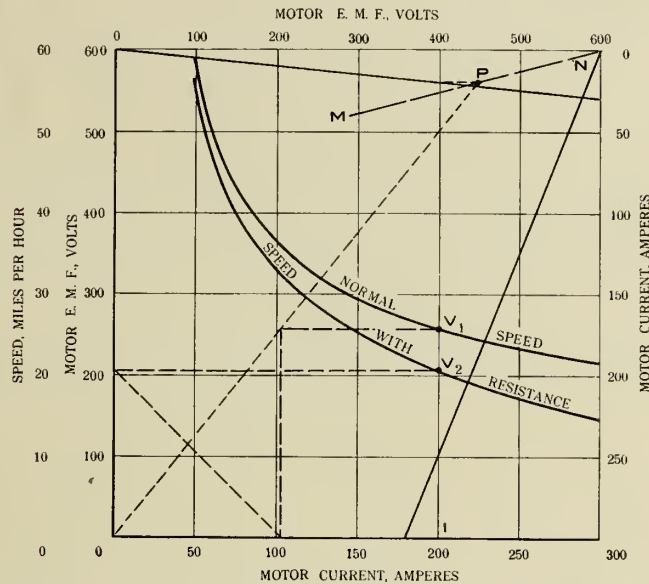


FIG. 5. SECOND METHOD FOR DETERMINING MOTOR SPEEDS WITH RESISTANCE.

is not the same, the line  $MN$  has a different angle, which is determined by the relative values of resistance in the two cases; that is, if the line  $MN$  of Fig. 5 makes an angle  $\theta$  with the axis of abscissae,

$$\tan \theta = \frac{r}{R + r} \dots \dots \dots (3)$$

With this modification the method is precisely the same as that described above.

#### IV. STARTING RESISTANCE FOR SERIES MOTORS WITH RHEOSTATIC CONTROL

In starting direct-current series motors it is usually not sufficient to reduce the potential at the motor terminals by making different combinations of motors on the supply circuit. When this can be done, as may be possible with very small motors, the performance may be predicted by calculating the performance curves at the lower potentials, as described previously in this bulletin, or by any other ordinary method. In general, however, it is necessary to place a certain external resistance in the circuit, whether or not the potential





at the terminals is reduced by any other means. The added resistance should be just sufficient to give the desired values of starting current and torque, the one being dependent on the other. As the motor gains speed, the resistance must be reduced, or the current and the torque will fall off too much. Of course, unless the resistance can be cut out in infinitesimal steps, there will be some variation in these quantities, the range being determined by the allowable difference between the maximum and minimum values of torque and current.

The simplest method of control consists merely in connecting the motor or motors to the line with an external resistance in series, the latter being reduced in steps until finally it is all out of the circuit and the motors are directly across the line. It is essential to determine correctly the exact values of resistance to be placed in circuit on each point of the controller in order that the conditions of current and torque limits may be met. This can be done quickly and accurately by a graphical method based on those given above.

When the motor is stationary the current which will flow is determined entirely by the resistances in the circuit, since the effect of inductance enters only at the instant of connecting to the line, and there is no counter e.m.f. being developed at the time. Since the internal resistance of a well-designed machine is quite small, it is necessary to add a considerable external resistance to keep the initial current down to a proper amount. The exact value of current desired depends on the torque needed and on the capacity of the motor and the connecting wiring. Having determined the required current, it is a simple matter to find the necessary resistance. This may be done directly by the application of Ohm's law. Let  $I_m$  be the maximum allowable motor current,  $E$  the line e.m.f.,  $r$  the motor resistance, and  $R_1$  the external resistance to be inserted at starting. Then

$$I_m = \frac{E}{R_1 + r} \dots \dots \dots (4)$$

from which  $R_1$  may be found at once if the other quantities are known. As soon as current flows through the motor, a torque is developed, and the armature will commence to rotate. This will cause the generation of a counter e.m.f. tending to oppose the e.m.f. of the circuit, so that the current will be reduced. The torque falls off correspondingly, and if the action is allowed to continue the performance will be as shown in Fig. 4 or Fig. 5, the acceleration dropping until the motor operates at some constant speed. Since it is usually desirable to bring the motor up to full speed as soon as practicable, it is customary to reduce the amount of resistance in the circuit so that the accelerating current will remain near the maximum value. The amount of resistance which should be removed from the circuit at one time is a function of the total number of steps in which it is to be cut out or the allowable variation from the mean value of the starting torque. The latter is the simpler case and will be considered first.





Assume that the allowable variation from the mean value of starting torque to give smooth acceleration is 10 per cent. The minimum torque will then be approximately 20 per cent less than the maximum, which latter corresponds to the current at standstill, as determined by equation (4). As previously explained, the current will decrease from the instant of starting until it has fallen to the minimum desired value determined from the proper acceleration. At this point the counter e.m.f. developed by the armature will have risen to some value which can be determined readily, since the sum of the resistance drop,  $I(R_1 + r)$ , and the counter e.m.f.,  $E_c$ , must equal the line pressure; that is,

$$E = E_c + I(R_1 + r) \dots\dots\dots (5)$$

Since the value of resistance has already been found by equation (4), the value of  $E_c$  can be obtained.

When the current has fallen to its minimum value  $I_n$  the resistance of the circuit should be reduced enough to bring the current up to the maximum value  $I_m$ . In order to find this new value of resistance, it is necessary to determine the counter e.m.f. which will exist after the change in connections has been made. If the field flux of the motor remained constant, then, disregarding small variations due to changes in armature reaction and other causes, the counter e.m.f. would be the same for any value of armature current. But in the series motor the field flux is a function of the armature current\*, since the latter also flows through the field. The flux will therefore become greater when the current is increased by the removal of some of the resistance. The exact amount of this change depends on the proportions of the magnetic circuits of the motor, and can be determined from the saturation curve of the machine. For practical purposes of calculating starting resistance, this method is not available, since it requires making a special test of the motor. There are, however, methods which may be used for getting approximate proportional values of flux which will serve the purpose equally well.

The speed of an electric motor varies directly with the counter e.m.f. developed and inversely with the field flux. From this it may be seen that the flux is directly proportional to the counter e.m.f. and inversely proportional to the speed; that is,

$$\Phi = \frac{E_c}{kn} \dots\dots\dots (6)$$

in which  $\Phi$  is the field flux,  $n$  the speed of rotation, and  $k$  a constant depending on the winding, etc.

Since

$$E_c = E - Ir \dots\dots\dots (7)$$

equation (6) may be rewritten

\*In case the field of a series motor is shunted, the current through it is directly proportional to that through the armature, although not equal to it.



$$\Phi = \frac{E - Ir}{kn} \dots\dots\dots (8)$$

In the ordinary motor it is not possible to determine with any accuracy the constant  $k$  unless access may be had to the design data. But, since in the calculation of starting resistances only *proportional* values of flux are required, the knowledge of this constant is entirely unnecessary. Therefore, the following equation may be used with equal accuracy:

$$k\Phi = \frac{E - Ir}{n} \dots\dots\dots (9)$$

If the motor resistance is known, the relation of  $k\Phi$  to the armature current  $I$  may be calculated for any current and a curve plotted if desired.

Another method of getting proportional values of flux depends on the relation of this quantity to the torque developed by the motor. In any electric motor the torque is proportional to the armature current and the field flux; that is,

$$D = K\Phi I \dots\dots\dots (10)$$

where  $D$  is the torque of the motor at a current  $I$ , and  $K$  is a proportionality constant depending on the winding, but not the same as  $k$  in the preceding equation. As before, a curve may be plotted, giving proportional values of flux for any armature current.

When the current is increased from  $I_n$  to  $I_m$  by reducing the resistance in the circuit, the flux increases from  $\Phi_n$  to  $\Phi_m$ . During the infinitesimal time required for changing the current, it is evident that the speed cannot change. It must follow, therefore, that the counter e.m.f. will increase, due to the greater flux. By equation (5), the new value will be the counter e.m.f.,  $E_{cm}$ , at the minimum current,  $I_n$ , multiplied by the ratio of fluxes. The new counter e.m.f.,  $E_{cm}$ , can then be found as follows:

$$E_{cm} = E_{cn} \left( \frac{k\Phi_m}{k\Phi_n} \right) \dots\dots\dots (11)$$

or,

$$E_{cm} = E_{cn} \left( \frac{K\Phi_m}{K\Phi_n} \right) \dots\dots\dots (12)$$

depending on which method was used for getting the proportional values of flux. For brevity, call this ratio of field fluxes  $Q$ ; that is,

$$Q = \frac{k\Phi_m}{k\Phi_n} = \frac{K\Phi_m}{K\Phi_n} \dots\dots\dots (13)$$

$$E_{cm} = QE_{cn} \dots\dots\dots (14)$$

Then,

If the maximum and minimum values of current are to be reached each time the resistance is changed, then the ratio  $Q$  becomes



constant for the particular conditions assumed, and the calculation of resistances is simplified considerably. On the other hand, it may be advisable to allow different values of current on the various steps of the controller, in which case the ratio of fluxes must be determined separately for each point. When the controller is equipped with a current-limiting device the former condition holds. By the application of the above equations the values of resistance for a rheostatic controller may be calculated.

It is more convenient for the engineer to calculate the resistances by a graphical process, since the use of the equations is somewhat tedious. For this purpose the volt-ampere diagram may be employed conveniently. In Fig. 6 the volt-ampere diagram of Fig. 1

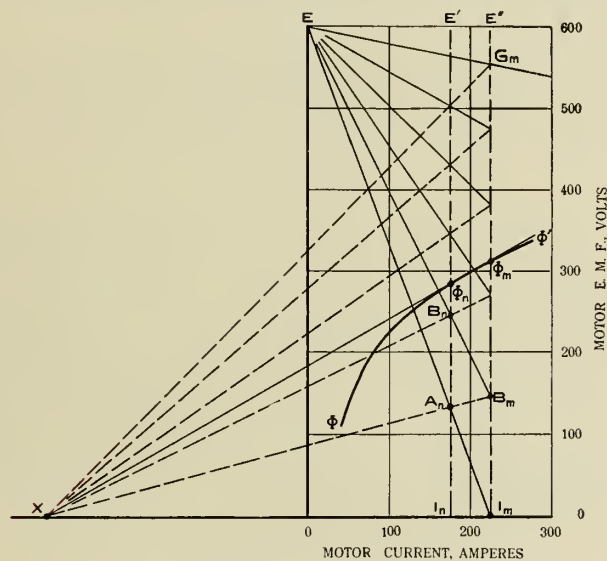


FIG. 6. DIAGRAM FOR DETERMINING RESISTANCES FOR SERIES MOTOR WITH RHEOSTATIC CONTROL.

has been repeated, and on it is also plotted the curve of relative values of flux ( $k\Phi$  or  $K\Phi$ ) against current. The limits  $I_m$  and  $I_n$  being chosen, it is evident that the ratio  $Q$  will be constant. If, then, a line is drawn through the points  $\Phi_m$  and  $\Phi_n$ , cutting the axis of abscissae at  $X$ , the latter will be the intercept of all lines cutting the verticals through  $I_m$  and  $I_n$  at points proportional to these values of flux; that is, in the figure,

$$\frac{I_m \Phi_m}{I_n \Phi_n} = \frac{I_m B_m}{I_n A_n} \text{ etc., } = Q \dots \dots \dots (15)$$

since all of the triangles whose apexes pass through the point  $X$  divide parallel lines into proportional parts.





Starting with the maximum current  $I_m$ , the entire external potential  $E$  is used up in overcoming resistances. That is, the line  $I_mE''$  represents the  $IR$  drop,  $I_mG_m$  being that in the external resistor and  $G_mE''$  that in the motor itself. As soon as the armature begins to rotate a counter e.m.f. is developed. When the motor current has fallen to  $I_n$  this e.m.f. is represented by the ordinate  $I_nA_n$ , the line  $I_mE$  being drawn through  $E$ , for evidently there will be no  $IR$  drop with zero current. It is evident that when the current has reached  $I_n$  resistance must be cut out in one step until the current rises to the maximum,  $I_m$ . Since the counter e.m.f. has a value of  $I_nA_n$  when the current is a minimum, it follows that it must increase by the ratio  $Q$  when the current is increased to  $I_m$  so rapidly that the motor does not have time to change its speed. The new counter e.m.f. may be determined by projecting a line from  $X$  through  $A_n$ , intersecting the line of maximum current at  $B_m$ . The counter e.m.f. at this point is represented by  $I_mB_m$ , the drop in the external resistor by  $B_mG_m$ , and that in the motor by  $G_mE''$ . The external resistance to be employed is found by dividing  $B_mG_m$  by the current  $I_m$ . The process may now be continued until all the external resistance has been removed and the motor is running on the line. This condition is shown by the line  $EG_m$ , and from this point on the normal curves of motor performance apply.

If it is desired to change the current limits at any stage of the controller operation, the proper resistance can be determined in the same manner, the location of the point  $X$  being varied to correspond to the proper values of current. For small changes, the location of  $X$  may be assumed constant without introducing an appreciable error. If a definite number of steps is called for, as by the adoption of a standard controller, the values of  $I_m$  and  $I_n$  must be changed until the exact number of steps is obtained on the diagram. This must be done by trial, but the adjustment can be made quickly after a few cases have been solved.

As given above, the diagram has been worked out for a single series motor. If two motors are to be run in parallel, it is only necessary to modify the diagram to give the proper values of current, remembering that the combined resistance of the machines is but one-half that of a single motor. For operation with machines in series the same precautions must be observed, but in this case the motor resistance is twice that of a single machine. With these variations, the diagram can be modified to meet any combinations of rheostatic control of series motors.

#### V. SERIES-PARALLEL CONTROL\*

In electric railway practice it is customary to operate series motors in pairs or in groups of motors in pairs. They are ordinarily

\*See Electric Railway Journal, Dec. 26, 1914, and Feb. 13, 1915.





controlled by the *series-parallel* method, which involves placing the two units in series with resistance which is cut out in steps, changing to parallel with the resistance again inserted, and finally cutting it out again in steps. Generally the current limits are the same for both connections, although sometimes they are different in the series and in the parallel arrangements.

The calculation of the counter e.m.f. and the resistance for series-parallel control is made in the same manner as for the rheostatic, except that the precautions mentioned under the former topic on p. 15 must be observed very carefully. It is usually convenient to combine the series and the parallel diagrams into one. This is shown in Fig. 7. The method of construction is the same as for

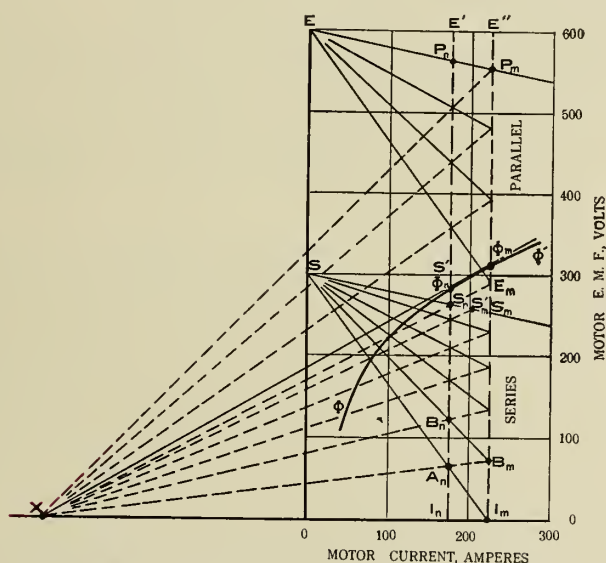


FIG. 7. DIAGRAM FOR DETERMINING RESISTANCES FOR SERIES MOTORS WITH SERIES-PARALLEL CONTROL.

rheostatic control, the difference being that the point  $S$ , representing half potential, is taken as the point for drawing the  $IR$  lines while the motors are in series, and the point  $E$  for the same purpose after the parallel connection is made. It is necessary to interpret correctly the values of  $IR$  drop to determine the resistances. When the motors are in series the current flowing through the circuit is that through a single machine, while after they are thrown in parallel the line current is that for two motors. To determine the series resistances, therefore, the external  $IR$  drop, for instance that on the first point of the controller, is equal to  $I_m S_m$  per motor, so that this



must be doubled to get the total drop in the external circuit. The correct value of resistance to put in series with the motors on the first point is then

$$R_1 = \frac{2I_m S_m}{I_m} \dots \dots \dots (16)$$

and similarly for any other value of series resistance.

When the connections are changed from series to parallel, the counter e.m.f. of each motor is  $I_n S_n$  just before breaking the circuit, and  $I_m E_m$  after the reconnection is complete. In series, the counter e.m.f.'s of the two motors add, while in parallel they do not. The residue,  $E_m P_m$ , must therefore be consumed in external resistance. On the first parallel point the resistance must then be

$$R_p = \frac{E_m P_m}{2I_m} \dots \dots \dots (17)$$

and so on until the motors are directly on the line. In all other respects the series-parallel diagram is precisely the same as the rheostatic diagram previously described.

#### VI. STARTING RESISTANCE FOR SHUNT MOTORS

The calculation of starting resistances for shunt motors is made in the same manner as for series machines, the principal difference

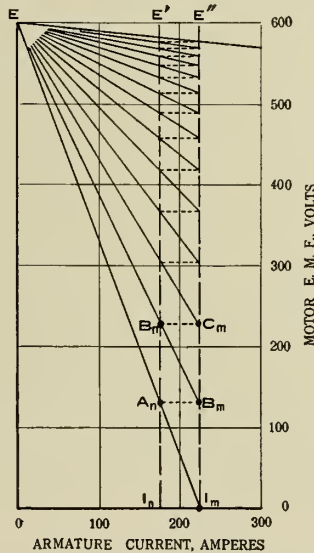
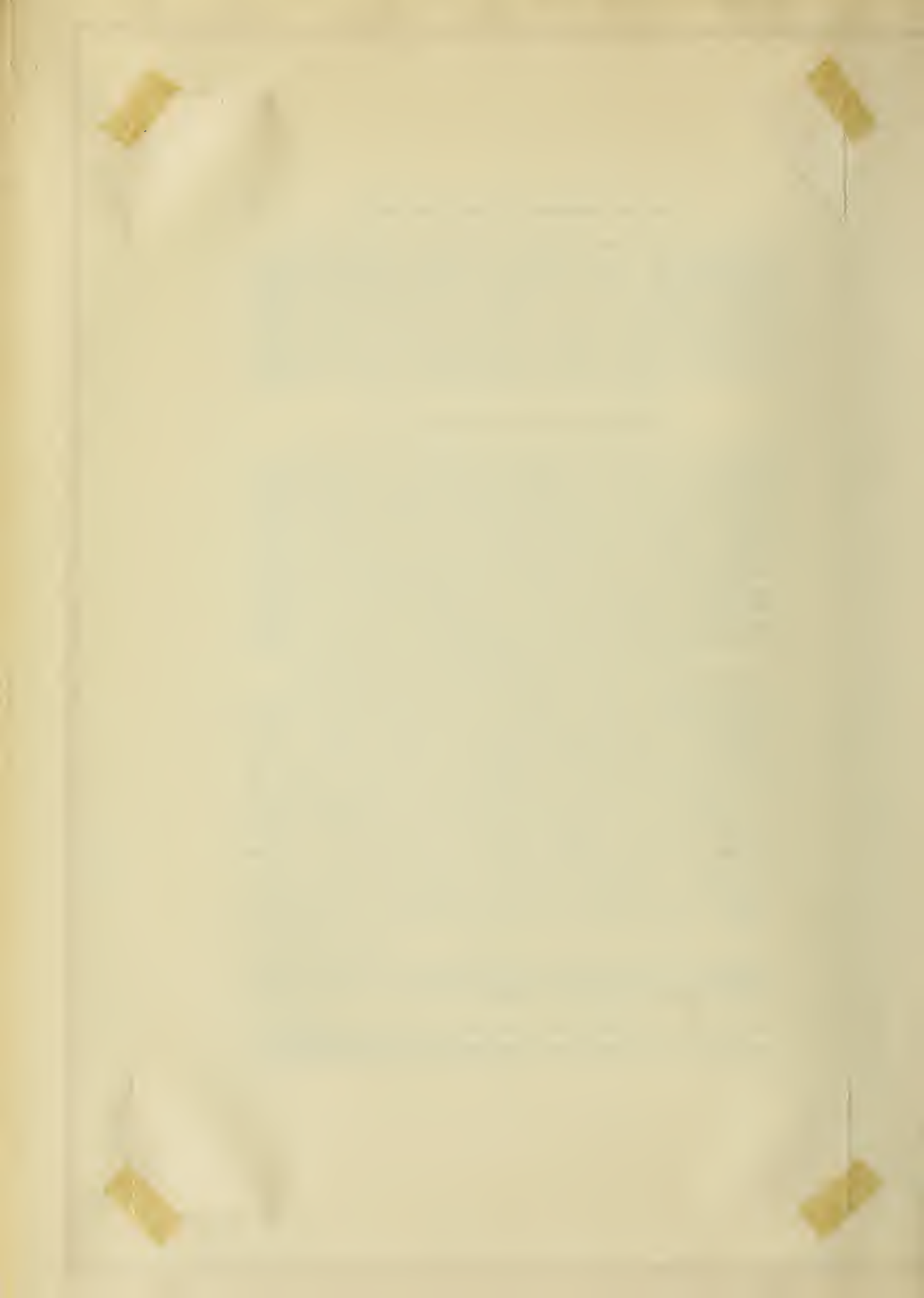


FIG. 8. DIAGRAM FOR DETERMINING RESISTANCES FOR SHUNT MOTOR WITH RHEOSTATIC CONTROL.

being that, since the field is supplied by a circuit in parallel with the armature, the field flux is practically constant at a given potential for all values of armature current. It is, therefore, unnecessary to determine any change of flux when the resistance is reduced.



The diagram for calculating graphically the values of armature resistance is given in Fig. 8 for a single motor. This diagram is somewhat similar to Fig. 6, except that the lines representing the change from one point to the next are not drawn through a single point *X*, but are all parallel to the base. The method of getting the resistances from measurements on the diagram is the same as previously described. For series-parallel control a similar scheme may be followed. It is not illustrated here on account of the infrequency of the use of series-parallel control with shunt motors.

### VII. PLOTTING SPEED-TIME CURVES

A number of methods have been proposed from time to time to reduce the labor incident to the plotting of speed-time curves for railway trains. The analytical solutions all depend on producing equations representing the characteristic curves of the motive power; and, on account of the difficulty of determining separately the equation of the curve for each separate motor or locomotive, general solutions giving the average of a large number of machines have been used. Although this is satisfactory for approximate calculations in which extreme accuracy is not required, as in preliminary estimates, it is not suitable for problems involving a particular machine. For such cases graphical or semi-graphical methods are usually resorted to if a solution more rapid and less laborious than that obtained by the point-by-point construction is desired.

Of the graphical methods, the first one which was satisfactory was that developed by Mr. C. O. Mailloux.\* The construction there described is of a high degree of accuracy, and is so simple that it may be readily applied. It has the disadvantage of requiring a number of charts on which the graphical solution is based, and which take considerable time for preparation. Although the method saves labor when a large number of determinations must be made for the same equipment, the time taken for construction of the charts is a serious disadvantage when but a few runs are to be calculated. A scheme intended to obviate the latter difficulty was devised by Professor E. C. Woodruff,† in which the separate charts are replaced by diagrams drawn directly on the motor curve-sheet. Although the work of plotting is somewhat less than in the Mailloux method, and the intermediate calculations are all on the single motor curve-sheet, considerable time is still required for plotting the diagrams needed in the determination.

From time to time constructions have been developed for accomplishing portions of the desired result, and these may be considered useful for modifications of the original methods just described. They

\*Notes on the Plotting of Speed-Time Curves, Transactions A. I. E. E., Vol. XIX, p. 901 (1902).

†Graphic Method for Speed-Time and Distance-Time Curves, Transactions A. I. E. E., Vol. XXXIII, p. 1673 (1914).





simplify and in some cases reduce the labor incident to the graphical calculation.

The plan herein proposed is a graphical solution which possesses the accuracy of the original ones, while at the same time it eliminates nearly all of the intermediate steps. The calculations are all based on fundamentally correct principles, and the results may be determined as closely as desired within the limits of accuracy of the ordinary methods of plotting.

The acceleration produced by a known tractive effort is given in the following equation:

$$A = \frac{F}{91.1(1+r)T} \dots \dots \dots (18)$$

in which  $A$  is the acceleration in miles per hour per second,  $F$  the *net* tractive effort of the motor in pounds at the wheel treads,  $T$  the weight of the train in *tons per motor*, 91.1 the force needed for unit acceleration of translation alone, and  $r$  the ratio of force required for the acceleration of rotating parts to that for translation. When extreme accuracy is not necessary, equation (18) can be replaced by the simpler statement

$$A = \frac{F}{100T} \dots \dots \dots (19)$$

in which the rotating parts are assumed to take approximately one-tenth the force necessary for acceleration of translation. It is evident from these equations that for a given weight of train per motor the acceleration produced is directly proportional to the net tractive effort.

The force available for acceleration, or net tractive effort, is the residue of the total torque of the motors, after reducing to the speed at the wheel treads, subtracting the force for overcoming train resistance and curve resistance, and subtracting or adding the force for going up or down grades. The size and type of the cars making up the train being known, and the profile given, it is a comparatively simple matter to determine these quantities. Train resistance may be calculated from tests or by any one of a number of well-known formulæ, as, for example, that developed by Mr. A. H. Armstrong:

$$R = \frac{50}{\sqrt{W}} + 0.03 V + \frac{0.002 a V^2}{W} \left( 1 + \frac{n-1}{10} \right) \dots \dots \dots (20)$$

in which  $R$  is the train resistance in pounds per ton,  $W$ , the weight of the train in tons,  $V$ , the train speed in miles per hour,  $a$ , the projected cross-section of the train, and  $n$ , the number of cars in the train. This is probably as accurate as any general equation developed for passenger cars. For freight trains, other equations should be used.\*

\*See Bulletin 43, Engineering Experiment Station, University of Illinois. "Freight Train Resistance," by Edward C. Schmidt.





Grades require an additional tractive effort of 20 pounds per ton for each per cent of up grade, and correspondingly less for down grade. Curve resistance is quite difficult to determine, but may be assumed from 0.5 pound to 2.0 pounds per ton per degree of curvature. After making the proper subtractions and additions to the gross tractive effort given by the motive power, the force available for producing acceleration, or net tractive effort, is obtained.

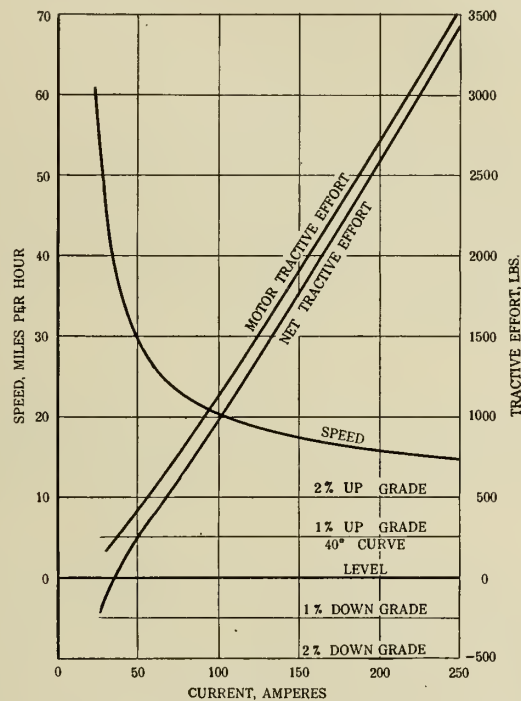


FIG. 9. RAILWAY MOTOR CHARACTERISTIC CURVES.

Since the mathematical operations for getting the net tractive effort are addition and subtraction, the calculation may be made by a graphical process. This has been explained by many previous writers, so that it is not necessary to repeat it here.\* It is worth noting, however, that if the train resistance is subtracted directly on the diagram, the residue represents at once the net tractive effort for level track, while if plotted separately the process of subtraction is rendered more difficult, requiring the use of a scale or a pair

\*C. O. Mailloux, Notes on the Plotting of Speed-Time Curves, Transactions A. I. E. E., Vol. XIX, p. 901 (1902).  
 E. C. Woodruff, Graphic Method for Speed-Time and Distance-Time Curves, Transactions A. I. E. E., Vol. XXXIII, p. 1673 (1914).  
 A. M. Buck, The Electric Railway, p. 39.



of dividers, in addition to the coordinate scales of the chart. Grade and curve resistances being single-value functions (i.e., not changing with speed), they may be represented by horizontal lines on the chart, either increasing or decreasing the ordinates of tractive effort.

The manufacturer's performance curves for a certain electric railway motor are given in Fig. 9, with the addition of the train resistance and grade and curve resistances for use with a particular train or car. The net tractive effort curve gives directly the accelerating force for level track; and for other conditions the base may be moved up or down as required. It is to be noted that the values of resistance are plotted in terms of force per motor, so that if, for example, the equipment consists of four motors, the values on the chart will be one-fourth of the total.

The net tractive effort having been determined, the acceleration produced may be found from equations (18) or (19). These equations show that if the tractive effort is plotted as an ordinate and the quantity  $100T$  from equation (19) as an abscissa, the slope of the line connecting the origin with the point thus determined is a measure of the acceleration to the same scale. The actual value of the slope is not important; it depends on the units chosen for the coordinates of the speed-time curve.

In plotting the speed-time curve, the most satisfactory way is to take an increment of speed,  $\Delta V$  and, knowing the value of acceleration,  $A$ , to determine the corresponding increment of time,  $\Delta t$ . It is this method which has been elaborated by all writers and which is the basis of the present article. Since

$$A = \frac{\Delta V}{\Delta t} \dots \dots \dots (21)$$

then

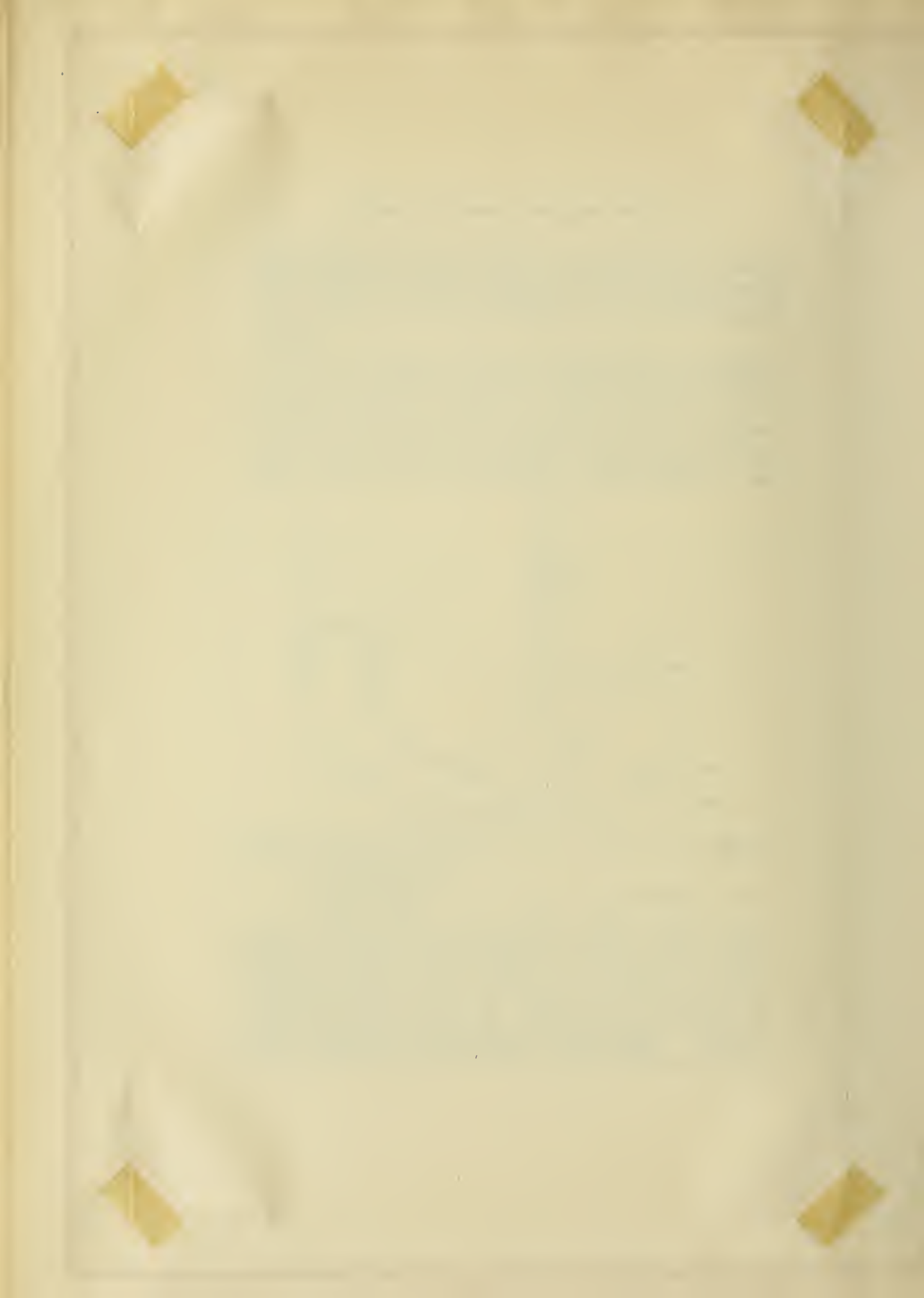
$$\Delta t = \Delta V \frac{1}{A} \dots \dots \dots (22)$$

This equation is the basis of the former methods of graphical determination of speed-time functions. In Mailloux' method, a chart of inverse values of  $A$  and of integral multiples of these values is plotted. An inspection of equation (22) shows that if an increment  $\Delta V$  equal to unity is taken,  $\Delta t$  is the reciprocal of  $A$ , so that it may be taken directly from the chart. A somewhat similar method is followed by Woodruff, who, however, combines the reciprocal curve and the chart of accelerations on one sheet.

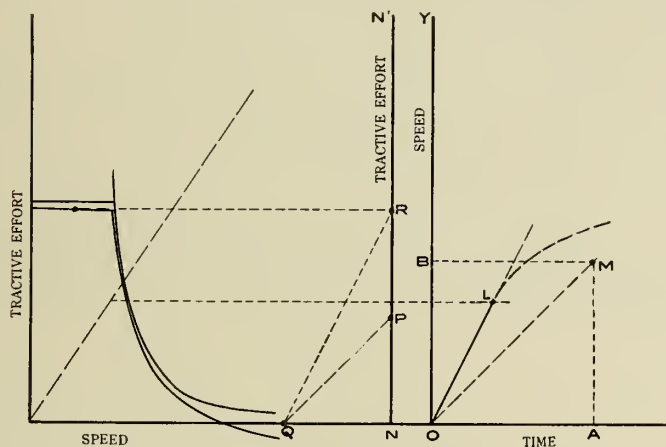
A comparison of equations (18) or (19) and (21) shows them to be of precisely the same form, so that they may be equated as follows:

$$\frac{\Delta V}{\Delta t} = \frac{F}{100T} \dots \dots \dots (23)$$

using the simpler form of the expression given in equation (19).



Let  $V_0$  be the scale of ordinates; namely, the number of miles per hour per unit of ordinates, and  $t_0$  the corresponding scale of abscissæ, as required for the speed-time curve. From equation (21) the slope of the acceleration line produced with these unit values may be determined. The scale for  $100T$  being arbitrary, then if it is chosen so that the slope of the line is the same as that for unit acceleration, it makes possible the direct construction of the speed-time curve. The diagram, Fig. 10, shows the arrangement. With ordinate  $OB$  and



abscissa  $OA$  on the speed-time curve, each equal to the unit selected. the corresponding acceleration is

$$\frac{OB \text{ (miles per hour)}}{OA \text{ (seconds)}} = A \text{ (miles per hour per second)} \dots (24)$$

The discussion shows that a definite amount of force,  $F_0$ , is required to produce this acceleration in a given weight of train. Selecting any suitable scale of tractive efforts, as  $NN'$ , this force may be represented by the ordinate  $NP$ . It is evident that if a straight line is drawn through  $P$  parallel to the acceleration line  $OM$ , cutting the horizontal axis at  $Q$ , the length  $NQ$  will represent the quantity  $100T$  to the proper scale. This is proved by equation (23) and the similarity of the triangles  $OMA$  and  $QPN$ . The same equation shows that any other value of tractive effort, as  $NR$ , will







produce an acceleration represented by the slope of the line  $QR$ , the corresponding location of the speed-time curve being  $OL$ , drawn with the same slope. The values of tractive effort for which the acceleration is determined may be read directly from the tractive effort curve, plotted against speed, as in Fig. 10, or from the curve of tractive effort plotted against current, as in Fig. 11.

The construction of the speed-time curve is now evident. If the train is started with any constant current,  $I_1$  (Fig. 11), the accelera-

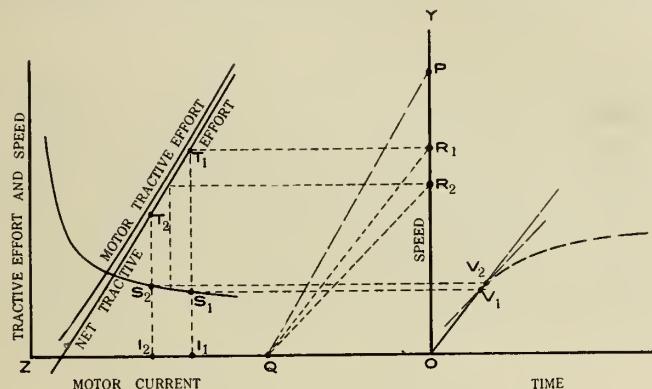


FIG. 11. CONSTRUCTION FOR PLOTTING SPEED-TIME CURVE FROM MOTOR SPEED-CURRENT AND TRACTIVE EFFORT-CURRENT CURVES.

tion produced is represented by the slope of  $QR_1$ . The line  $OV_1$  is then drawn with the same slope. The constant current can be maintained up to the speed  $S_1$ , so that the line  $OV_1$  should be continued until it reaches this ordinate at the point  $V_1$ . This line and point are on the speed-time curve to the desired scale. With further increase in speed, the tractive effort will decrease, as indicated by the curve, and the acceleration will be correspondingly less.

Consider an increment of speed,  $\Delta V = V_2 - V_1$ . This corresponds to a decrease of tractive effort from  $T_1$  to  $T_2$ . If the increment is taken small enough that the variation in force is practically along a straight line, the average tractive effort, acting continuously for a time  $\Delta t$ , will produce an increase in velocity  $\Delta V$ . If, then, the tractive effort at the mean speed,

$$V_1 + \frac{1}{2}\Delta V = \frac{1}{2} (V_1 + V_2) \dots\dots\dots (25)$$

is taken and projected on  $OY$ , at  $R_2$ , the slope of the line joining this point with  $Q$  is the average acceleration during the increment. A line drawn through  $V_1$  parallel to  $QR_2$  will pass through the point  $V_2$  at the end of the increment  $\Delta V$ . The location may be made conveniently by projecting  $S_2$  parallel to the axis of abscissæ, and noting the intersection  $V_2$ . If the increment has been taken sufficiently small, this is a point on the curve, and not a tangent; for the



tangent to the curve at the mean ordinate would not pass through the points  $V_1$  and  $V_2$ , but would be parallel to the line drawn through them. The magnitude of the error due to this assumption is fully discussed by Mailloux.\* It is shown that the error is so small as to be negligible in ordinary calculations if  $\Delta V$  is not too great.

The construction outlined in the last paragraph may now be continued for the remainder of the acceleration period until the train reaches constant speed. A smooth curve drawn through the points located in this manner is the true speed-time curve; and the accuracy may be made as great as desired by proper choice of the speed increments.

For the coasting portion of the curve, the train resistance may be plotted to any horizontal scale, the ordinates being the same as those for motor tractive effort. In fact, the ordinates representing train resistance, which are plotted down from the gross tractive effort curve, may be stepped off with dividers and transferred to the line  $OY$  to determine the corresponding retardation. Speed increments may be taken as before, and the coasting curve plotted. For the braking curve an ordinate corresponding to the braking force must be obtained and added to the train resistance. In this manner the entire speed-time curve may be determined.

#### VIII. PLOTTING DISTANCE-TIME CURVES†

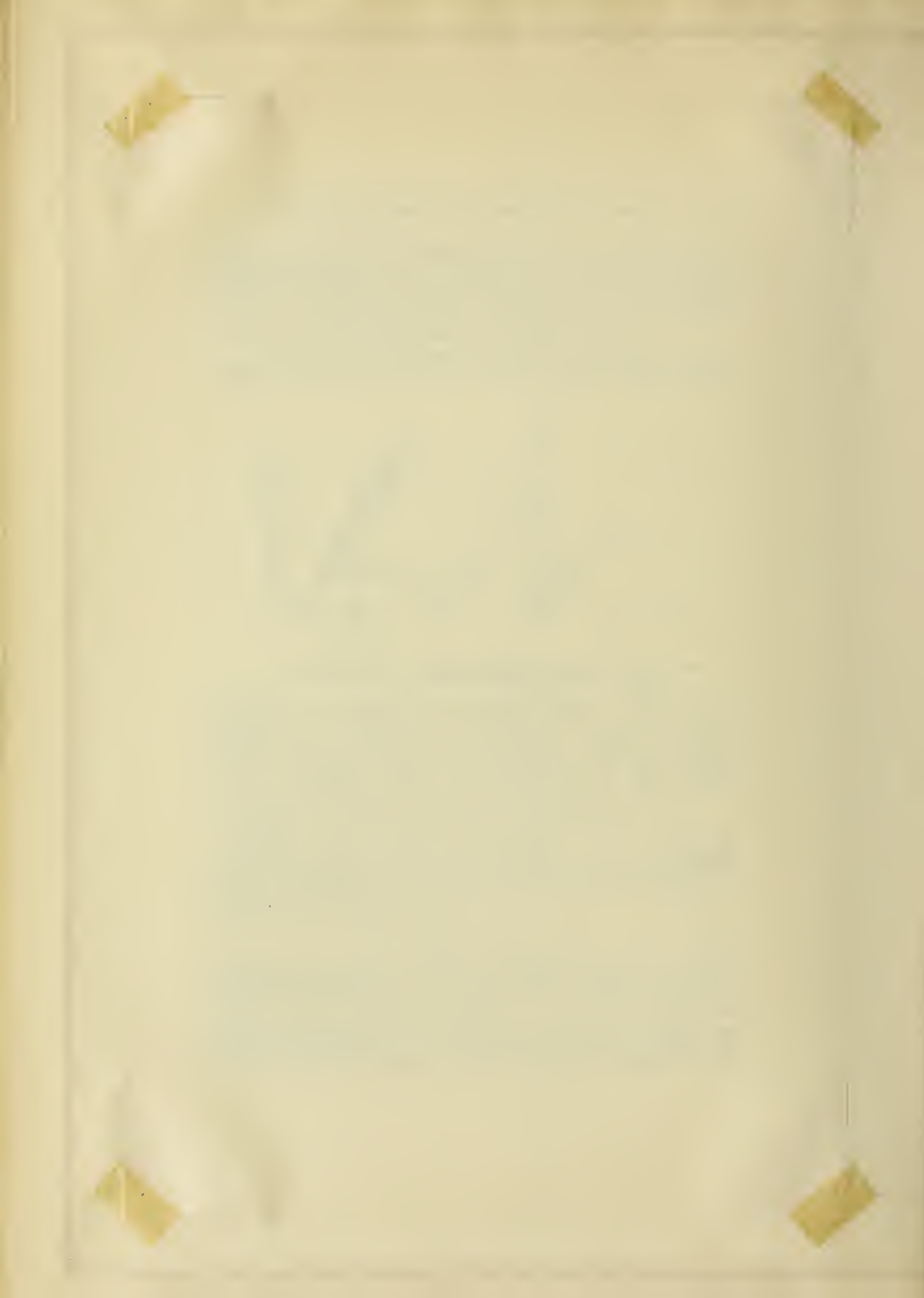
In constructing distance-time curves, a number of methods may be used. Mailloux determines distance by means of the device known as the "integraph," which is a convenient and accurate way. If such an instrument is not available, a planimeter may be used, making partial integrations over portions of the run, so that enough points may be located to draw the curve. This is a much slower process, although of practically the same accuracy as the former. In the absence of any other device, the area of the curve may be determined by making the plot on coordinate paper and counting the small squares included by the diagram. Woodruff uses a series of curves representing distance covered at average speeds, which may be used in estimating the distance passed over during the various increments.

A method which is at least as accurate as any of the purely graphical constructions mentioned is described in the following paragraph:

Assume any convenient scale of distance to be used for plotting the distance-time curve on the same sheet as the speed-time curve. Referring to Fig. 12, let  $OB$  represent unit distance, say one mile. This same ordinate corresponds to a speed of  $V$  miles per hour on

\*Transactions A. I. E. E., Vol. XIX, p. 988 (1902).

†The process described for plotting distance-time curves is a general method of graphical integration, and may be used for the construction of integral curves for any function whatever that may be represented by Cartesian graphs.



the speed-time curve. If the train continues in motion at a velocity of  $V$  miles per hour for  $\frac{1}{V}$  hours, the distance covered will evidently be one mile. Since the speed in such motion is constant, the rate of

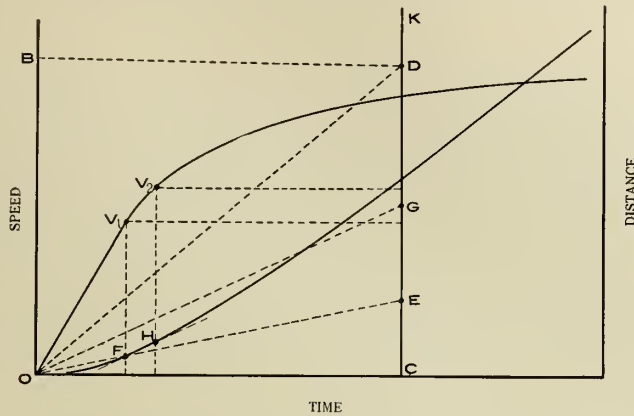
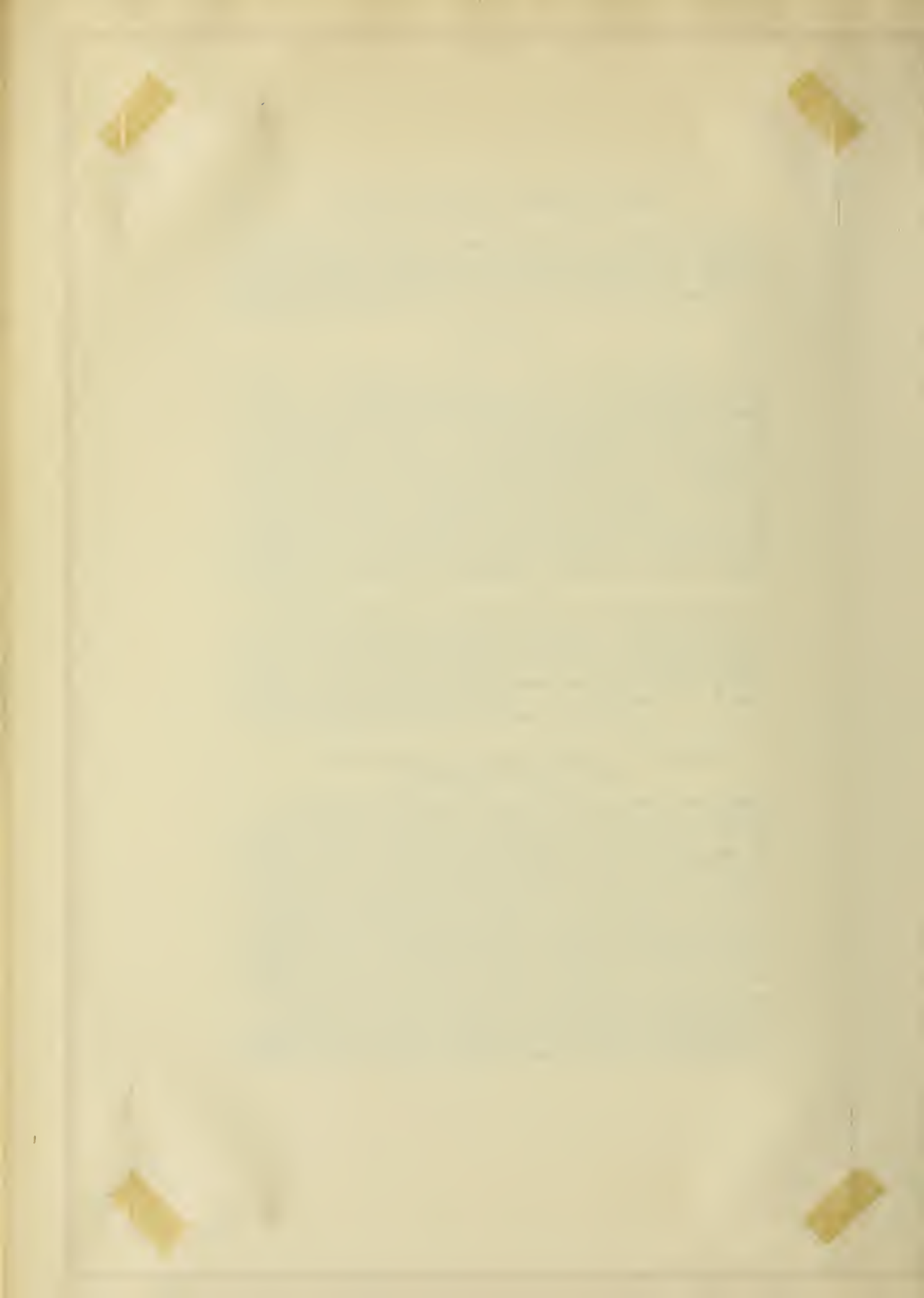


FIG. 12. CONSTRUCTION FOR PLOTTING DISTANCE-TIME CURVE FROM SPEED-TIME CURVE.

covering distance, or the slope of the distance-time curve representing the run, is a straight line. Lay off a length  $OC$  on the time axis equal to  $\frac{1}{V}$  hours, and erect the perpendicular  $CDK$  at  $C$ . A diagonal line connecting  $O$  and  $D$  will then measure the distance traversed when the speed is represented by the ordinate  $CD = OB$ . In other words,  $OD$  is the correct distance-time curve for a constant speed  $OB = V$ . For any other time, the distance covered will be proportional, and will be represented equally well by the ordinate of the line  $OD$  up to that time. Since distance is proportional to the product of speed and time, the distance covered at any other velocity during the time  $\frac{1}{V}$  hours will be represented by an ordinate equal to that speed.

This construction may be utilized in plotting the distance-time curve from the speed-time curve as follows. Take the *average* velocity during any time increment and project the ordinate representing it on the line  $CDK$ . The intercept on the line joining the projection of this average speed with the origin included within the limits of the time increment measures the distance covered. For instance, the first portion of the speed-time curve, terminating in the point  $V_1$ , has been made at a constant acceleration. The average speed during the







increment is  $\frac{1}{2}(V_1+0)$ . Locate the point  $E$  on the line  $CDK$  so that  $CE = \frac{1}{2}V_1$ . Connect  $O$  and  $E$  by the straight line  $OE$ . The time increment at the point  $V_1$  intersects this line at  $F$ . This is a point on the distance-time curve since, for uniform acceleration, the distance  $s$  is

$$s = \frac{V_0 + V_1}{2} \Delta t \dots\dots\dots (26)$$

which is a fundamental relation. For the next increment, from  $V_1$  to  $V_2$ , the average velocity,  $\frac{1}{2}(V_1 + V_2)$ , is represented by the ordinate  $CG$ , and the line  $OG$  determines the slope of the distance curve during this period. A line  $FH$ , beginning at the point  $F$  and drawn parallel to  $OG$  will, therefore, determine the point  $H$  on the distance curve at the end of the time increment. This construction may be continued until the entire distance-time curve is located. A smooth graph passing through the points thus plotted is the true distance-time curve. As in the case of the speed-time curve, the points located are actually on the curve and not on tangents. The construction is accurate so long as the deviation of the speed-time curve from a straight line is negligible during each increment under consideration.

It is not claimed that this method of determining distance is more accurate than the use of the integrator or the planimeter, but that it is of more ready application, and gives results which are as accurate as are ordinarily obtainable within the limitations of curve plotting. The error can be made as small as desired by taking increments of time of such magnitude that the speed-time curve is practically straight during any one of them, as explained.

#### IX. APPLICATIONS OF GRAPHICAL METHOD FOR SPEED-TIME AND DISTANCE-TIME CURVES

A problem frequently met with in railway service is the determination of the exact points of cutting off power and of applying brakes in order to make a run of fixed distance in a given time. The solution may be made by the application of the speed-time and distance-time curves. To do this the braking portion of the speed-time curve may be plotted backward from the end of the run and the corresponding distance curve located, while the distance curve for acceleration is plotted forward from the zero point. A period of coasting must be interposed which will satisfy the operating requirements; namely, one which will allow braking to be included at the normal rate and also reach the desired point for the end of the run.

In order to show the method, a complete speed-time and distance-time curve will be drawn. While the entire plot is given for straight level track, the modifications for various combinations of grade and curve may be made as suggested in the foregoing para-



graph. A diagram drawn by this method is shown in Fig. 13. The run comprises an acceleration with the motors, followed by a period of coasting, and lastly by a period of braking. This is the simplest form of run ordinarily used, although it is possible to eliminate the coasting, applying the brakes immediately after cutting off the power.

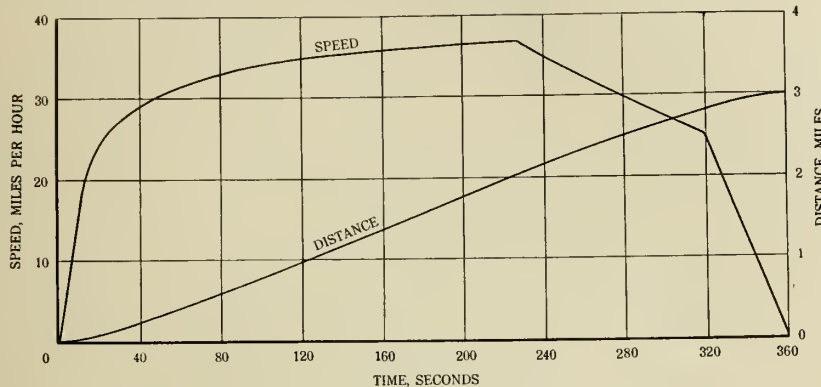


FIG. 13. COMPLETE SPEED-TIME AND DISTANCE-TIME CURVES.

The latter practice, however, is not usual. The various parts of the run are determined independently and afterward connected together as indicated in the following paragraphs. The plot of Fig. 13 checks with that produced by an analytical determination within the limits of accuracy of the cross-section paper used; and the graphical construction has the further advantage of requiring only a set of triangles or a parallel ruler when the same scale of ordinates is used for the speed-time curve as that given on the motor characteristic curve.\*

The braking rate is usually assumed constant. A speed-time curve for this portion of the run may be plotted backward from the end, as in Fig. 14, and the corresponding distance-time curve determined.

The coasting speed-time curve is independent of the acceleration and the braking, for during this period the train is acted on solely by the force of train resistance and the incidental resistances present due to the track conditions. For a given profile the coasting speed-time curve may be determined from the weight of the equipment and the train resistance equation. It may be drawn graphically by the methods of Fig. 10 or Fig. 11, the motor tractive effort being replaced by the train resistance per motor (i.e., the total resistance per train

\*It is often undesirable to plot the speed-time curve to the same speed scale as that of the motor performance. In such a case the time corresponding to a certain increment of speed may be found directly by laying off a right triangle, the hypotenuse of which is parallel to the acceleration line. Since this triangle may be plotted to any scale whatever, the accuracy may be as great as desired. From the successive speed and time increments thus found, a speed-time curve may be plotted. The distance-time curve may be laid out in a similar manner.



divided by the number of motors). It should be remarked that a resistance is a negative force, and should, therefore, be plotted downward from the base. The acceleration produced will be negative unless the force due to a down grade is such as to equal or exceed

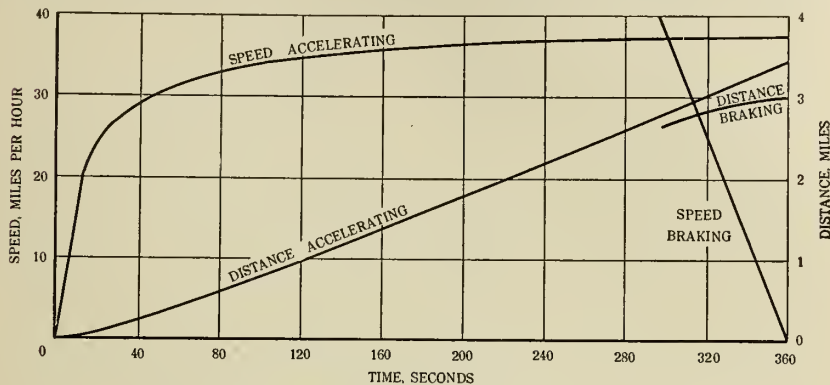


FIG. 14. METHOD OF DETERMINING PROPER POINT FOR CUTTING OFF POWER.

the negative force of train resistance. A separate speed-time curve for coasting may be plotted on tracing paper or other transparent medium and the corresponding distance-time curve located, as shown in Fig. 15.

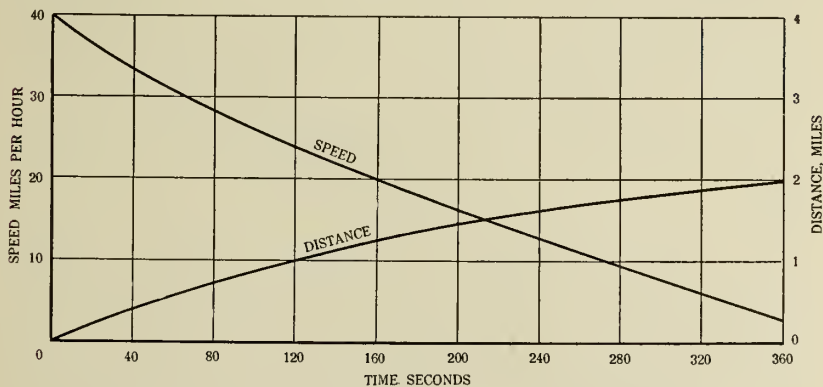


FIG. 15. COASTING SPEED-TIME AND DISTANCE-TIME CURVES.

Since the distance-time curve is the first integral of the speed-time curve, an abrupt change in the slope of the latter corresponds to a point of inflection in the former, or merely to a change in its curvature. The coasting distance-time curve must, therefore, be tangent to both the accelerating and the braking portions of the







distance-time curve for the run. For further proof of this it may be noted that, since the *slope* of the distance-time curve is a measure of the speed, this slope must be the same for either curve at the point where the two portions of the speed-time curve join. This fact makes possible the following method of accurately locating the points of cut-off of the current and application of the brakes.

The tracing of the coasting distance curve (Fig. 15) should be laid over the curve of distance while accelerating (Fig. 14) with the axes of coordinate parallel, so that the two curves are tangent at some point. The tracing should then be slid along, *keeping the axes parallel*, until the coasting curve also becomes tangent to the braking distance curve. The points of tangency thus determined correspond to the cut-off of the current and the application of the brakes. These points having been determined and the distance-time curve during coasting transferred to the plot of Fig. 14, the tracing of the coasting curves may be moved *parallel to the axis of ordinates* until the two axes of abscissæ coincide. The coasting line may now be traced on Fig. 14, locating definitely the remainder of the speed-time curve and producing the complete diagram of Fig. 13.

In practice, it is usually convenient to have a number of coasting curves, corresponding to different conditions of grade and track curvature, to cover all the variations liable to occur. Such a series, plotted on a sheet of tracing cloth or transparent celluloid, forms a templet for the location of the principal points on the speed-time and distance-time curves for runs of definite length, making the graphical construction of much greater value in preliminary calculations to determine the size of motors required for a given service.

The graphical method of plotting speed-time and distance-time curves described is equally good for use with any kind of motive power. All that is necessary is to get the relation between speed and tractive effort connected by a graph which can then be used for determining accelerations in the manner outlined. The application is so obvious that it need not be further elaborated.

#### X. HEATING VALUE OF A VARIABLE CURRENT.

The rating of all electrical apparatus depends to a considerable degree on the heating of the active parts. This is especially true in the case of railway motors. One of the principal sources of heating is the resistance of the conductors. The heat produced in a wire carrying a current is proportional to the square of the current multiplied by the time during which it is acting. In general, the value of current in a conductor is not fixed for any considerable period, but is constantly changing. If the variation follows some known law, the effect of the current in producing heat can be found by a comparatively simple mathematical analysis; but if the current is changing in some casual or variable way, the evaluation is not easy.



The latter condition holds true in the case of the electric railway motor cycle. Here the current is a maximum at the instant of starting, after which it gradually falls to a minimum, and is then cut off entirely while the train coasts and comes to rest. The variation is further complicated on account of the occurrence of grades, curves, points where the speed must be reduced, and other special conditions of operation.

Railway motors are usually rated by the current which can be carried continuously, or for a stated period, with a temperature rise above the surrounding air considered safe.\* To determine whether or not a motor is large enough for a given service, the variable current must be evaluated to find whether it is above or below a safe amount. The method usually employed is to plot the curve of current taken by the motor against time and from this construct another curve of values of current squared. The integral of the latter curve, divided by the total time of operation, is the square of that current which, applied continuously for the same time, will produce the same loss in the conductors.

As ordinarily applied this method is cumbersome. It requires the use of a table of squares or some similar method of calculation, so that the new curve can be plotted from the original current values. To obviate the necessity of squaring a large number of values, another plan has been devised, which requires the replotting of the current curve in polar coordinates. The effective current can be obtained by this method without the need of squaring the ordinates of the current curve.†

The entire argument in favor of the use of the polar diagram for finding the effective motor current is that it is less laborious than to plot the curve of squared values of current. Two methods, both of them entirely graphical, will now be described for plotting the latter curve, which is more easily prepared by these methods than the polar diagram. The other operations involved in the determination of the effective current are essentially the same for either this or the polar method. The curve of squares of current plotted on a rectangular base has the further advantage that it can be put on the same sheet with the original current curve, thus rendering unnecessary the use of a separate chart and making possible an easier coordination of the values than when the diagrams employed are so different in character as the rectangular and the polar graphs.

In Fig. 16 consider a scale of natural numbers,  $ON$ . Corresponding to these it is desired to construct another scale of natural numbers such that a certain ordinate  $O'M$  is equivalent to  $\overline{ON}^2$ . It is evident that the square of  $ON$  may be represented by the area

\*For further information regarding the methods in vogue for rating railway motors, see Standardization Rules of the A. I. E. E., 1915 edition.

†For a proof, see A. M. Buck, The Electric Railway, p. 136.





$ONBA$  enclosed by the rectangle having each of its sides equal to  $ON$ . Since the scale of squares is chosen so that  $O'M$  is numerically equal to the square of  $ON$ , it may equally well be stated that it represents the area  $ONBA$ . The problem is to find the ordinate along

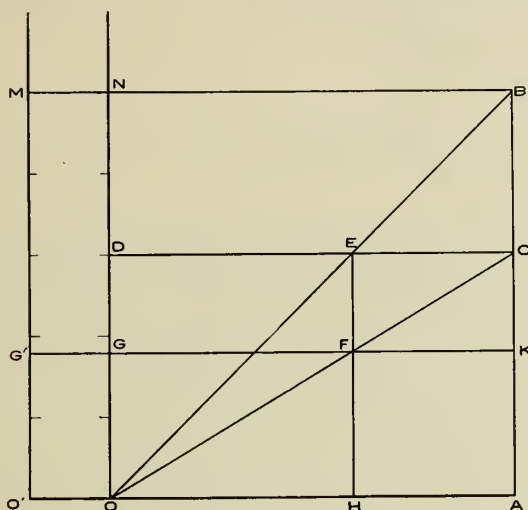


FIG. 16. METHOD FOR SQUARING NUMBERS GRAPHICALLY.

$O'M$  corresponding to the square of some other value, as  $OD$  on the original scale. It has been seen that the scale  $O'M$  may be considered to measure areas, so that the discussion resolves itself into finding the ordinate along  $O'M$  which will represent the area  $ODEH$ , which is the square constructed on the side  $OD$ . If a rectangle with the base  $OA$  can be found with an equivalent area, its ordinate will be the value sought.

Referring to Fig. 16, construct the diagonal  $OB$  of the large square, and continue  $DE$  to meet  $AB$  at the point  $C$ . Connect  $C$  with  $O$ , cutting  $HE$  at  $F$ . The geometrical construction gives

$$ON = AB = OA = NB \dots\dots\dots (27)$$

$$OD = HE = OH = DE = AC \dots\dots\dots (28)$$

$$OG = HF = AK \dots \dots \dots (29)$$

$$\frac{HE}{AB} = \frac{AC}{AB} = \frac{OD}{ON} \dots\dots\dots (30)$$

$$\frac{HF}{HE} = \frac{AC}{AB} = \frac{OD}{ON} \dots\dots\dots (31)$$

Hence

$$OG = HE \times \frac{OD}{ON} \frac{AC}{AB} = OD \times \dots\dots\dots (32)$$





Therefore

$$\text{Area } OGKA = \text{Area } ODEH \dots\dots\dots (33)$$

Since

$$\begin{aligned} \text{Area } OGKA &= OG \times OA \\ &= \left( OD \times \frac{OD}{ON} \right) \times ON \\ &= \overline{OD} \dots\dots\dots (34) \end{aligned}$$

Hence  $O'G'$ , the numerical equivalent of  $OG$ , represents  $\overline{OD}^2$  to the scale of  $O'M$ .

The application of the method is obvious. It is only necessary to construct a square at any point on the current-time chart, with a side such that some value of current and its square are represented by the same length of side. Any value of current, corresponding to  $OD$ , should be projected on the diagonal  $OB$  and also on the side  $AB$  of the square. When the projection  $AC$  on  $AB$  has the point  $C$  connected with  $O$ , the line  $CO$  will intersect  $HE$  in the point  $F$ . This is the ordinate for the curve of current squared, and may be carried back to the proper position above or below the corresponding value of current. With a small amount of practice the calculation can be made with great rapidity, for the construction lines can be very largely omitted, only the intersections being required to find the proper values.

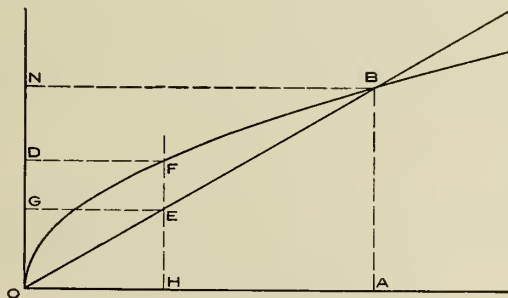


FIG. 17. PARABOLIC CURVE FOR SQUARING NUMBERS.

An alternative method to that just described is to plot a curve between the natural numbers and their squares, the latter values being represented by convenient ordinates. An inspection of Fig. 16 indicates that the locus of the points  $F$  is a parabola whose principal axis is  $ON$  and which passes through the point  $B$ . In practice it is found simpler to make the diagram of the opposite form, as shown in Fig. 17. The parabola  $OFB$  is of the form

$$x = k y^2 \dots\dots\dots (35)$$



Consider a line  $OEB$  drawn through the origin. Ordinates cut off by this line, as  $HE$ , are proportional to the abscissæ, as  $OH$ . Corresponding ordinates on the parabola are proportional to the square roots of the abscissæ. Therefore  $HE$  is equal to  $\sqrt{HF}$  on such a scale that  $AB$  is represented by the same ordinate,  $AB$ . The construction holds for any line  $OB$  intersecting the parabola.

In order to apply the method just described, the parabola  $OFB$  and the straight line  $OEB$  should be plotted on some transparent medium, such as celluloid. The templet thus made may be slid along the curve of current with the axis  $OA$  coinciding with the base line of the current curve, until the parabola intersects the current curve at the proper point; the square of that ordinate will then be found directly under or over the value of current. This can be repeated an indefinite number of times until sufficient points are obtained to plot the curve of current squared. From this the effective current may be obtained, as explained above.

Since the plotting of points as obtained by the parabolic curve may be difficult when the base lines  $OA$  coincide, since holes will have to be pricked through the templet, the method may be modified to permit the construction being placed on an ordinary celluloid triangle by moving the axis of the line  $OEB$  upward through a suitable distance. This is shown in Fig. 18. Here the base line

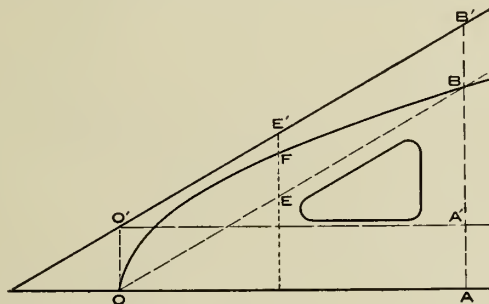


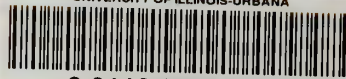
FIG. 18. APPLICATION OF PARABOLIC CURVE TO A TRIANGLE.

for the parabola is  $OA$ , the edge of the triangle; while that for the diagonal line has been transferred to  $O'A'$ , at a distance  $OO'$  above the other axis. All the ordinates along  $O'E'B'$  are therefore displaced by the amount  $O'B'$ . This will not occasion any difficulty in the subsequent calculations, since the value obtained for the area of the current squared curve will be too great by an amount equal to  $OO'$  multiplied by the length of the diagram. As the area is to be divided by the base to find the mean ordinate, the calculation can be made without reference to the constant, and the value of  $OO'$  subtracted from the mean ordinate for the current squared curve.





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